Stating the Main Theorem

Yesterday, Claudia ended by sketching a functor

$$\mathrm{Spf} : \left(\mathrm{CAlg}^{\mathrm{aug}}_{\mathbb{K}}\right)^{op} \longrightarrow \mathrm{Fun}\left(\mathrm{CAlg}^{\mathrm{sm}}_{\mathbb{K}}, \mathcal{S}\right)$$

There are two versions

① Via 1-categories equipped with weak equivalences

not sure how to distinguish notationally is a nice way: I will underline for today

$\mathcal{S} = \underline{\mathrm{sSet}}$ w/ wk. htpy eq [model cat.]

$\underline{\mathrm{CAlg}^{dg}}$ = comm dg algs over $\mathbb{K}$ concentrated in degrees ≤ 0 w/ quasi-iso [model cat]

$\underline{\mathrm{CAlg}^{\cancel{dg}, \mathrm{aug}}} = \left(\mathrm{CAlg}^{dg}\right)_{/\mathbb{K}} = \begin{cases} \text{obj} : & A \to \mathbb{K} \\ \text{mor} & A \to B \text{ over } \mathbb{K} \end{cases}$

[w/ inherited model str]

$\underline{\mathrm{CAlg}^{\cancel{dg}, \mathrm{small}}}$ = full subcategory of $\mathrm{CAlg}^{dg, \mathrm{aug}}$ of small algebras [no model cat]

$\underline{\mathrm{Fun}(\mathrm{CAlg}^{\cancel{dg}, \mathrm{small}}, \mathrm{sSet})}$ = functors where $X \xrightarrow[\sim]{f} Y$ if $X(A) \xrightarrow[\sim]{f(A)} Y(A)$ $\forall A \to \mathbb{K}$ [no model cat]

Examples are often "presented" in this setting

② Via ∞-categories (i.e. nice version like quasicategories)

$\mathcal{S}$ = $(\infty,1)$-category equivalent (as $(\infty,1)$-cats!) to "sSet" or "Top"

$\mathcal{C}\mathrm{Alg}$ = $(\infty,1)$-category presented by CAlg^{dg}

(e.g. $N(\underbrace{\mathrm{CAlg}^{dg\,\circ}}_{\text{subcat of fibrant \& cofibrant objects}})$)

CAlg^{aug} = slice $(\infty,1)$-category

CAlg^{sm} = full subcategory of small algebras

$\mathrm{Fun}(\mathrm{CAlg}^{sm}, \mathcal{S})$ = $(\infty,1)$-category of functors

Theorems are easy to prove here.

__Remk__: One role for the abstract machinery is to relate these two settings.

In ②:

$$\mathrm{Spf}\begin{pmatrix} A \\ \varepsilon_A \downarrow \\ \mathbb{K} \end{pmatrix}\begin{pmatrix} R \\ \varepsilon_R \downarrow \\ \mathbb{K} \end{pmatrix} = \mathrm{Map}_{\mathrm{CAlg}^{aug}}\left(\begin{matrix} A \\ \varepsilon_A \downarrow \\ \mathbb{K} \end{matrix}, \begin{matrix} R \\ \varepsilon_R \downarrow \\ \mathbb{K} \end{matrix}\right)$$

pullback in $\mathcal{S}$ of the diagram

$$\begin{tikzcd} & \mathrm{Map}_{\mathrm{CAlg}}(A,R) \arrow[d, "\varepsilon_R \circ -"] \\ * \arrow[r, "\varepsilon_A"] & \mathrm{Map}_{\mathrm{CAlg}}(A,\mathbb{K}) \end{tikzcd}$$

[need to talk about pullbacks!]

MT2

Let me mention that you can compute this in 1-categorical setting:

$$\mathrm{Map}_{\mathrm{CAlg}}(A,R) \simeq \Big(\overbrace{[n] \longmapsto \mathrm{CAlg}\big(A, R\otimes \underbrace{\Omega^*_{\mathrm{poly}}(\Delta^n)}\big)}^{\text{simplicial set}}\Big)$$

$$\mathbb{K}[\underbrace{t_0,\dots,t_n}_{0}, \underbrace{dt_0,\dots,dt_n}_{1}] \Big/ \left(\begin{array}{l}\sum t_i = 1,\\ \sum dt_i = 0\end{array}\right)$$

w/ $\partial(t_i) = dt_i$

"polynomial de Rham forms on n-simplex Δ^n"

As usual, you should choose A, R sufficiently nice (e.g. cofibrant & fibrant, respectively)

In a moment, we will discuss pullbacks, but if you know homotopy pullbacks of top. spaces, you can mimic that here.

We also have a functor

$$C^*_{\mathrm{Lie}} : \mathrm{Lie}^{dg}_{\mathbb{K}} \longrightarrow (\mathrm{CAlg}^{dg,\mathrm{aug}}_{\mathbb{K}})^{op}$$
$$(\mathfrak{g},\partial) \longmapsto C^*_{\mathrm{Lie}}(\mathfrak{g}) = \left(\mathrm{Sym}_{\mathbb{K}}(\mathfrak{g}^*[-1]),\ d_{CE} + \partial\right)$$

augmentation is projection onto $\mathrm{Sym}^0 \cong \mathbb{K}$

that preserves weak equivalences

$\Rightarrow \quad C^*_{\mathrm{Lie}} : \mathrm{Lie}_{\mathbb{K}} \longrightarrow (\mathrm{CAlg}^{\mathrm{aug}}_{\mathbb{K}})^{op}$

a functor of $(\infty,1)$-categories

We can now state the main theorem.

<u>Theorem</u>

The composite functor

$$\Psi = \mathrm{Spf} \circ C^*_{\mathrm{Lie}} : \mathrm{Lie}_{\mathbb{k}} \longrightarrow \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{sm}}_{\mathbb{k}}, \mathcal{S})$$

factors through $\mathrm{FMP} \subseteq$ "

and is an equivalence of ∞-categories

$$\mathrm{Lie}_{\mathbb{k}} \xrightarrow{\Psi} \mathrm{FMP}$$

<u>recall</u>: $\mathrm{Ho}(\Psi)$ is an equivalence

& $\Psi_{g,h} : \mathrm{Map}_{\mathrm{Lie}}(g,h) \xrightarrow{\simeq} \mathrm{Map}_{\mathrm{FMP}}(\Psi g, \Psi h)$

~~is a weak equivalence of simplicial sets~~ $\forall g, h$

To understand this functor, we certainly need to understand homotopy pullbacks better! That will be a goal of the rest of today.

We will use our improved understanding to construct the "tangent functor"

$$T : \mathrm{FMP} \longrightarrow \mathrm{Mod}_{\mathbb{k}} \quad (\text{coming from}^{\text{the model cat.}}\ \mathrm{dgVect}_{\mathbb{k}})$$

which computes the <u>tangent complex</u> of a moduli functor & generalizes the usual notion of a ~~[illegible]~~ Zariski tangent space. It will "linearize" our situation & be the first step towards an inverse functor.

Pullbacks from the perspective of ∞-categories

There are different perspectives useful for different situations

- descriptions by universal property: good for proving structural theorems! e.g. in quasicategories
- descriptions by construction: good for building stuff! e.g. in model categories

v1

Here is a clean abstract definition.

Let $\mathcal{C}, \mathcal{D}$ be quasicategories

Let $F: \mathcal{D} \to \mathcal{C}$ be a functor, i.e., map of simplicial sets.

Let $\mathcal{D}^{\triangleleft}$ be $\mathcal{D}$ with an "initial ∞-object attached",

i.e., $\Delta^0 \star \mathcal{D}$ $\quad \left((\Delta^0 \star \mathcal{D})_n = * \cup \mathcal{D}_n \cup \bigcup_{i+j=n-1} \mathcal{D}_j \right)$

Def A <u>limit of F</u> is a functor $\overline{F}: \mathcal{D}^{\triangleleft} \to \mathcal{C}$ such that ① $\overline{F}\big|_{\mathcal{D}} = F$ ~~and~~

② $\overline{F}$ is final among such extensions.

for any extension G of F to $\mathcal{D}^{\triangleleft}$, the space of maps $\{G \to \overline{F}\}$ is contractible

Another picture

$$\begin{tikzcd}
\mathcal{C}_{/F} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{p.b. in sSet}"] & \mathrm{Fun}(D^{\triangleleft}, \mathcal{C}) \arrow[d, "-|_D"] \\
* \arrow[r, "F"'] & \mathrm{Fun}(D, \mathcal{C})
\end{tikzcd}$$

I mean here the simplicial set of such functors

this is again a quasicategory & you can ask for a final object

This notion certainly captures the idea of a limit, and the use of sSet keeps track of homotopy coherence automatically.

But it may not be easy to exhibit a particular limit in practice.

v2 Suppose we have a 1-category $\mathcal{C}$ with a subcategory of weak equivalences W.

Let D be another category

Then

$\mathrm{Fun}(D, \mathcal{C}) = \{$1-category of functors $D \xrightarrow{X} \mathcal{C}\}$

inherits a natural notion of weak equivalence:

$X \xrightarrow[\simeq]{f} Y$ ← natural transformation   means $\forall d \in D$, $f(d) : X(d) \xrightarrow{\simeq} Y(d)$ in C.

i.e. $\in W$

Ex:
$$\begin{array}{ccc} & & A \\ & & \downarrow \varphi \\ 0 & \longrightarrow & B \end{array}$$
a diagram in dgVect, so φ is a chain map

This diagram is weakly equivalent to

$$\begin{array}{ccc} & & A \\ & & \downarrow \varphi \\ \mathrm{Cone}(id_B) & \xrightarrow{\simeq} & B \end{array}$$

← could also add an acyclic complex here

Their pushouts are not the same!

$\mathrm{Cone}(id_B)_n = B_n \oplus B_{n+1} \longrightarrow B_n$

$(b; b') \longmapsto b$

$d_{cone}(b, b') = (d_B b, b + (-1)^n d_B b')$

$H_*(\mathrm{Cone}(id_B)) = 0$

since $d_{cone}(b, b') = 0 \Leftrightarrow b$ closed & exact w/ $d_B b' = \pm b$

& then $d_{cone}(b', 0) = (d_B b', b')$

fix ordering

Ex $* \leftarrow S^{n-1} \rightarrow *$ & $D^n \leftarrow S^{n-1} \rightarrow D^n$ are weakly equivalent but their pushouts are not the same: $* \not\simeq S^n$!

We thus have a functor

$$\begin{array}{ccc} Ho(\mathrm{Fun}(D, C)) & \xleftarrow{\Delta} & Ho(C) \\ \Downarrow & & \Downarrow \\ (d \overset{\Delta_c}{\longmapsto} c) & \longleftarrow & [c] \\ \forall d \in D & & \end{array}$$

Def A homotopy limit is a right adjoint of Δ:

$$\mathrm{Ho\text{-}lim}_D : Ho(\mathrm{Fun}(D, C)) \longrightarrow Ho(C)$$

In other words,

$$\mathrm{Ho}(\mathcal{C})\Big([c], \text{Ho-lim}([F])\Big) \cong \mathrm{Ho}(\mathrm{Fun}(D,\mathcal{C}))\Big([\Delta_c], [F]\Big)$$

This is the "cone definition" again but for all D-diagrams at the same time "global"

When $\mathcal{C}$ is a model category, one can construct a representative of Ho-lim(F) in the functor category itself (fine print: there are some technical conditions on $\mathcal{C}$ and/or D to make sure you can do this but for pullbacks, things are ok)

Ex $D = \{a \to b \leftarrow c\}$ so

$$\mathrm{Fun}(D, \mathcal{C}) = \left\{ \begin{array}{ccc} & & X(c) \\ & & \downarrow \\ X(a) & \to & X(b) \end{array} \right\}$$

Prop This functor category admits a model structure where $f : X \to Y$ is

- a weak equivalence if f is an objectwise weak equivalence in $\mathcal{C}$

$f(a) : X(a) \xrightarrow{\simeq} Y(a)$ ...
in $\mathcal{C}$

- a <u>cofibration</u> if f is an objectwise cofibration

- a <u>fibration</u> if

  1) $f(b) : X(b) \longrightarrow Y(b)$ is a fibration

  2) the canonical map

$$\begin{tikzcd} X(a) \arrow[dr] & & \\ & Y(a) \times_{Y(b)} X(b) \arrow[r] \arrow[d] & X(a) \arrow[d] \\ & Y(a) \arrow[r] & Y(b) \end{tikzcd}$$

  is a fibration

  3) likewise $X(c) \longrightarrow Y(c) \times_{Y(b)} X(b)$ is a fibration

Argument is straightforward: restriction to b is just C & then one "extends" to a & c. Try, e.g., lifting properties.

<u>Prop</u> The functor

$$\Delta : C \longrightarrow \mathrm{Fun}(D, C)$$
$$x \longmapsto (a, b, c \overset{\Delta_x}{\longmapsto} x)$$

preserves cofibrations & acyclic cofibrations (and wk equivs)

~~[illegible] right Quillen~~

so its right adjoint $\lim_D$ is Quillen.

<u>Def</u> The <u>homotopy pullback</u> of $F \in \mathrm{Fun}(D, C)$ is $\mathbb{R}\lim_D (F)$, so can be computed by replacing F by a fibrant diagram

Extended example

deg 0

Let's prove, as a key example, that $\mathbb{K}[x]/(x^3)$ is a homotopy pullback of the diagram

$$\begin{tikzcd} & \mathbb{K} \arrow[d, "g"] \\ \mathbb{K}[x]/(x^2) \arrow[r, "f"] & \mathbb{K}[\varepsilon_{-1}] \end{tikzcd} \qquad (\star)$$

deg -1

This is a model example of an <u>elementary extension</u>.

<u>Basic procedure</u>:

- find a "fibrant replacement" $(\widetilde{\star})$ the diagram $(\star)$
- compute the ordinary ("naive") pullback of the diagram $(\widetilde{\star})$

(i) <u>Recall</u> in $\mathrm{CAlg}^{dg}_{\mathbb{K}}$, a map $A \xrightarrow{\varphi} B$ of cdgas is

- a <u>weak eq</u> if φ is quasi-isomorphism
- a <u>fibration</u> if φ is degreewise surjective

Every cdga is thus fibrant, so $\mathbb{K}[\varepsilon_{-1}]$ is fibrant

(ii) Let's replace f by a fibration:

$$\begin{array}{cccc} R := \left(\mathbb{K}[\overset{0}{x}, \overset{-1}{\delta}],\ \partial : \delta \mapsto x^2\right) & & x & \delta \\ \tilde{f} \big\downarrow & & \big\downarrow & \big\downarrow \\ \mathbb{K}[\varepsilon_{-1}] & & 0 & \varepsilon_{-1} \end{array}$$

$\partial(x^n\delta) = x^{n+2} \Rightarrow \ker \partial = 0$

& $\operatorname{coker} \partial = \mathbb{K}[x]/(x^2)$

$\Rightarrow R \simeq \mathbb{K}[x]/(x^2)$

By construction, $\tilde{f}$ is a levelwise surjection

(iii) Let's replace g_0 by a fibration:

$$\begin{array}{cccc} R' = \left(\mathbb{K}\overset{-1}{\delta} \oplus_{\mathbb{K}} \mathbb{K}\overset{0}{y},\ \partial'(\delta) = y\right) \ \&\ y^2 = 0 = y\delta & & y & \delta \\ \tilde{g} \big\downarrow & & \big\downarrow & \big\downarrow \\ \mathbb{K}[\varepsilon_{-1}] & & 0 & \varepsilon_{-1} \end{array}$$

(iv) What is the ordinary pullback

$$S = R \times_{\mathbb{K}[\varepsilon_{-1}]} R' \qquad ?$$

In degree 0, we have

$$\begin{tikzcd} & \mathbb{K} \oplus \mathbb{K}y \arrow[d] \\ \mathbb{K}[x] \arrow[r] & \mathbb{K} \end{tikzcd} \qquad \begin{tikzcd} y \arrow[d, mapsto] \\ 0 \end{tikzcd}$$

$$x \longmapsto 0$$

so $S^0 = \mathbb{K}[x] \oplus \mathbb{K}y$ where $x \cdot y = 0$

$$\left(\mathbb{K}[x] \times_{\mathbb{K}} (\mathbb{K} \oplus \mathbb{K}y) \right) \cong \mathbb{K}[x,y] / (y^2, xy)$$

In degree 1, we have

$$\begin{tikzcd} & \mathbb{K}\delta \arrow[d] \\ \mathbb{K}[x]\delta \arrow[r] & \mathbb{K}\varepsilon_{-1} \end{tikzcd}$$

so $S^1 = \mathbb{K}[x]\delta$

$\longrightarrow$: $x^n \delta \longmapsto \begin{cases} \delta & \text{if } n = 0 \\ 0 & \text{else} \end{cases}$

$\downarrow$: $x^n \delta \longmapsto x^n \delta$

And S has differential

$$d(\delta) = \partial(\delta) + \partial'(\delta)$$

$$= x^2 + y$$

(PB8)

$$n \geq 1: \quad d(x^n\delta) = \partial(x^n\delta) + \cancel{\partial'(x^n\delta)}^{\,0}$$

$$= x^{n+2}$$

hence $\quad \operatorname{coker} d \cong \mathbb{K}[x,y] \big/ (x^3, xy, x^2+y, y^2)$

$$\cong \mathbb{K}[x]/(x^3)$$

Discuss the generalizations: $\mathbb{K}[x]/(x^n)$ & elementary extensions

$$\begin{tikzcd} B \arrow[r] \arrow[d, "\sigma"'] \arrow[dr, phantom, "\text{h.p.b.}"] & \mathbb{K} \arrow[d] & \mathbb{K}\otimes\left(\underset{\deg -n}{\mathbb{K}\delta} \xrightarrow{id} \underset{\deg 1-n}{\mathbb{K}y}\right) \arrow[l, dashed, "\simeq"'] \arrow[dl] \\ A \arrow[r] & \mathbb{K}[\Sigma_{-n}] & \end{tikzcd}$$

Concluding remarks

Computing homotopy pullbacks can be hard work!

Computing homotopy limits for complicated diagrams is usually awful!

(But this can be true of ordinary limits)

Some tricks can help

① Fact if $\underline{X(a)} \xrightarrow{\sim} \underline{X(b)} \longleftarrow \underline{X(c)}$,

fibration

cofibrant

then naive pullback is a homotopy pullback

You don't need to replace both maps by fibrations

This fact implies

aka homotopy kernel of ϕ
or " fiber " "

Cor For $\phi : A \to B$ in dgVect, the homotopy pullback of $0 \longrightarrow B \xleftarrow{\phi} A$ is presented by

$$\begin{array}{ccc} \mathrm{Cone}(\phi)^n = (A^n \oplus B^{n+1}) & & (a, b) \\ \partial \downarrow & & \big\downarrow \\ \mathrm{Cone}(\phi)^{n+1} & & (d_A a,\ \phi(a) + (-1)^n d_B b) \\ \downarrow & & \\ \vdots & & \end{array}$$

Note. $\mathrm{Cone}(\phi) \longrightarrow A$ and there is a
$(a, b) \longmapsto a$

classical homological algebra coincides w/ "homotopical" algebra!

LES in $\cdots \to H^n \mathrm{Cone}(\phi) \to H^n A \longrightarrow H^n B \to H^{n+1} \mathrm{Cone}(\phi) \to \cdots$

Pf replace 0 by $\mathrm{Cone}(id_B) \simeq 0$ & take naive pullback as $\mathrm{Cone}(id_B) \to B$ is fibration $\square$

② The forgetful functor $\mathrm{CAlg}^{dg}_{k} \xrightarrow{fgt} \mathrm{dgVect}_k$ preserves limits & is a right Quillen functor so you can compute homotopy limits in dgVect, which is often "more obvious"

17510